EHRENFEUCHT-FRAÏSSÉ GAMES

Nicole Schweikardt
Institute for Computer Science
Johann Wolfgang Goethe-Universität Frankfurt am Main
Robert-Mayer-Str. 11–15, D-60325 Frankfurt am Main, Germany
schweika@informatik.uni-frankfurt.de

SYNONYMS

Ehrenfeucht games, EF-games

DEFINITION

The Ehrenfeucht-Fraïssé game (EF-game, for short) is played by two players, usually called the *spoiler* and the *duplicator* (in the literature, the two players are sometimes also called *Samson* and *Delilah* or, simply, *player I* and *player II*). The board of the game consists of two structures $\mathcal{A}$ and $\mathcal{B}$ of the same vocabulary. The spoiler's intention is to show a difference between the two structures, while the duplicator tries to make them look alike.
The rules of the classical EF-game are as follows: The players play a certain number r of rounds. Each round i consists of two steps. First, the spoiler chooses either an element a_i in the universe of $\mathcal{A}$ or an element b_i in the universe of $\mathcal{B}$. Afterwards, the duplicator chooses an element in the other structure, i.e., she chooses an element b_i in the universe of $\mathcal{B}$ if the spoiler's move was in $\mathcal{A}$, respectively, an element a_i in the universe of $\mathcal{A}$ if the spoiler's move was in $\mathcal{B}$.
After r rounds, the game finishes with elements $a_1, \ldots, a_r$ chosen in $\mathcal{A}$ and $b_1, \ldots, b_r$ chosen in $\mathcal{B}$, and exactly one of the two players has won the game. Roughly speaking, the duplicator has won if and only if the structures $\mathcal{A}$ and $\mathcal{B}$, restricted to the elements chosen during the rounds of the game, are indistinguishable. To give a precise description of the winning condition let us assume, for simplicity, that the vocabulary of the structures $\mathcal{A}$ and $\mathcal{B}$ only contains relation symbols. Precisely, the duplicator has won the game if and only if the following two conditions are met: (1) for all $i, j \in \{1, \ldots, r\}$, $a_i = a_j$ iff $b_i = b_j$, and (2) for each arity k, each relation symbol R of arity k in the vocabulary, and all $i_1, \ldots, i_k \in \{1, \ldots, r\}$, the tuple $(a_{i_1}, \ldots, a_{i_k})$ belongs to the interpretation of R in the structure $\mathcal{A}$ if and only if the tuple $(b_{i_1}, \ldots, b_{i_k})$ belongs to the interpretation of R in the structure $\mathcal{B}$.
Since the game is finite, one of the two players must have a *winning strategy*, i.e., he or she can always win the game, no matter how the other player plays.

MAIN TEXT

EF-games are a tool for proving expressivity bounds for query languages. They were introduced by Ehrenfeucht [1] and Fraïssé [3]. The fundamental use of the game comes from the fact that it characterizes first-order logic as follows: The duplicator has a winning strategy in the r-round EF-game on two structures $\mathcal{A}$ and $\mathcal{B}$ of the same vocabulary if, and only if, $\mathcal{A}$ and $\mathcal{B}$ satisfy the same first-order sentences of quantifier rank at most r (recall that the quantifier rank of a first-order formula is the maximum nesting depth of quantifiers occurring in the formula). This is known as the *Ehrenfeucht-Fraïssé Theorem*, and it gives rise to the following methodology for proving inexpressibility results, i.e., for proving that certain Boolean queries cannot be expressed in first-oder logic: To show that a Boolean query Q is *not* definable in first-order logic, it suffices to find, for each positive integer r, two structures $\mathcal{A}_r$ and $\mathcal{B}_r$ such that (1) $\mathcal{A}_r$ satisfies query Q, (2) $\mathcal{B}_r$ does not satisfy query Q, and (3) the duplicator has a winning strategy in the r-round EF-game on $\mathcal{A}_r$ and $\mathcal{B}_r$.
Using this methodology, one can prove, for example, that none of the following queries is definable in first-order logic: "Does the given structure's universe have even cardinality?", "Is the given graph connected?", "Is the given graph a tree?" (cf., e.g. the textbook [4]).
In fact, the described methodology is the major tool available for proving inexpressibility results when restricting attention to *finite* structures. Applying it, however, requires finding a winning strategy for the duplicator in the EF-game, and this often is a non-trivial task that involves complicated combinatorial arguments. Fortunately,

techniques are known that simplify this task, among them a number of sufficient conditions (e.g., *Hanf-locality* and *Gaifman-locality*) that guarantee the existence of a winning strategy for the duplicator (see e.g. the survey [2] and the textbook [4]).
Variants of EF-games exist also for other logics than first-order logic, e.g., for finite variable logics and for monadic second-order logic (details can be found in the textbook [4]).

CROSS REFERENCE

Locality, Logical Structure, Expressiveness of Query Languages, First-Order Logic

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