ZERO-ONE-LAWS

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SYNONYMS

zero-one law; 0-1 law

DEFINITION

A query language is said to have the *0-1 law* if every Boolean query that contains no constants (i.e., the query does not mention any particular element from the domain of potential data values) is almost surely true or almost surely false. The notions of being "almost surely true", respectively, "almost surely false" are defined as follows: Let σ be a fixed database schema. For each natural number n, let $DB_n(\sigma)$ be the set of all database instances of schema σ whose active domain is a subset of $\{1, \ldots, n\}$ (i.e., all database entries belong to $\{1, \ldots, n\}$). For a Boolean query q of schema σ let $\mu_n(q)$ be the probability that a database D chosen uniformly at random from $DB_n(\sigma)$ is a "yes"-instance of query q. I.e., $\mu_n(q)$ is the number of databases in $DB_n(\sigma)$ on which q evaluates to "yes", divided by the number of all databases in $DB_n(\sigma)$. Query q is said to be *almost surely true* (respectively, almost surely false), if the limit $\mu(q) := \lim_{n\to\infty} \mu_n(q)$ exists and is equal to 1 (respectively, 0).

MAIN TEXT

0-1 laws can be used as a tool for proving expressivity bounds for query languages: If q is a Boolean query for which the limit $\mu(q) = \lim_{n\to\infty} \mu_n(q)$ either does not exist or is different from 0 and 1, then q cannot be expressed by any query language that has the 0-1 law.

For example, let σ be the schema consisting of one binary relation symbol E, and let q_{even} be the query "Does the given database contain an even number of tuples?". It is not difficult to see that the limit $\mu(q_{even}) = \lim_{n\to\infty} \mu_n(q_{even})$ exists and is equal to 1/2. Thus, query q_{even} is neither "almost surely true" nor "almost surely false" and hence cannot be expressed by a query language that has the 0-1 law.

Historically, the first query language for which a 0-1 law was proved was the relational calculus (i.e., first-order logic). The proof also shows that there exists an algorithm which, given a Boolean relational calculus query q, computes $\mu(q)$. Since then, 0-1 laws have been shown for many different query languages, among them fixed point logic, infinitary logic $L^{\omega}_{\infty\omega}$, and various fragments of second-order logic. Variants of the 0-1 law are also known for several other probability measures and for restrictions to particular classes of databases.

A common method for proving that a query language has the 0-1 law is based on so-called *extension axioms* and an *Ehrenfeucht-Fraïssé game* argument. 0-1 laws are also closely related to the theory of the *countable random graph*. For an overview of results and proof techniques we refer to the textbooks [1,2,4] and the survey [3].

To point out the limitations of the use of 0-1 laws for proving inexpressibility results, it should be noted that there do exist queries that are almost surely true but nevertheless are not expressible in the relational calculus (an example is the query q_{conn} : "Does the given database relation E form a connected graph?").

Furthermore, several query languages are known *not* to have the 0-1 law, e.g., existential second-order logic and monadic second-order logic (cf. [1,3,4]).

CROSS REFERENCE

Expressiveness of Query Languages

REFERENCES

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